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Modified Supersymmetries

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An attempt is made to modify super-symmetry groups and the algebra of the underlying elements, so that the mappings involved in the realizations are those of an ordinary manifold on itself. The resulting group possesses representations that transform constant tensors and spinors among each other. Applications to fiber bundles lead to fields over general space–time manifolds whose structure group is the modified super-symmetry group.

1. INTRODUCTION

Super symmetries are characterized by graded Lie algebras, which differ from ordinary Lie algebras in that some elements of the algebra satisfy anti-commutation rather than commutation relations. Ordinary and graded Lie algebras have in common that they are linear vector spaces of a bilinear mapping that assigns to an ordered pair of elements another element. Whereas in an ordinary Lie algebra we have for such ordered pairs always the relationship

$$(a, b) = -(b, a) = c \tag{1.1}$$

in a graded Lie algebra some of these relationships are replaced by

$$(a, b) = (b, a) = c \tag{1.2}$$

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In addition to this relationship there exists an enveloping algebra with a different bilinear relationship, the product, which obeys the associative law

$$(ab)c \equiv a(bc) \tag{1.3}$$

Ordinary Lie algebras have been of interest to physicists because of their close relationship to infinitesimal groups, the corresponding finite groups representing some symmetry of the physical theory, i.e., some rule by which a possible history of a physical system is mapped on another possible history. Symmetries permit the arrangement of all possible histories into equivalence classes, such that the members of a given equivalence class can be carried over into each other by the action of the symmetry group. Presumably, graded Lie algebras can be interpreted as representations of an underlying symmetry.

Super symmetries and their graded Lie algebras include as an (ungraded) subalgebra the Poincaré Lie algebra, or some similar Lie algebra, such as the infinitesimal conformal group. Typically, the super symmetries combine in their transformation laws quantities with integral spin and quantities with half-odd spin. It is the latter that satisfy anticommutation relations among themselves, and the anticommutator between two "super-gauge transformations" is an element of the infinitesimal Poincaré group, a translation.

All the quantities with half-odd spin are assumed to be composed of "anticommuting numbers," i.e., of elements of a Grassmann algebra. Their product is assumed to satisfy the associative law, but also to be antisymmetric in its factors,

$$ab + ba = 0 \tag{1.4}$$

This assumption is perhaps suggested by the experience with the field quantization of fermion fields. Suppose i is an index characterizing a certain state of a single fermion, then the annihilation operators a_i form a set of q numbers whose product is skew symmetric,

$$a_i a_j = -a_j a_i \qquad (a_i)^2 = 0 \tag{1.5}$$

Since these quantities are linear operators defined on the vector space of sets, they form an associative algebra. In spite of these excellent heuristic credentials, the assumption that the spinors are composed of elements of a Grassmann algebra leads to some undesirable results, which have been described, for instance, by Corwin et al. (1975). Specifically, the "translation," the group-theoretical commutator (algebraic anticommutator) of two super-gauge transformations, though it commutes with all the components of the fermion variables, differs from ordinary complex quantities in that it is a null divisor, and in fact nilpotent.

It is the purpose of this paper to explore alternative algebraic constructions, which can serve as the realization of symmetries similar to those postulated in the literature on super symmetries while preserving the space–time coordinates as ordinary numbers. The principal novel point of view is that the anticommuting numbers will be replaced by elements of a linear vector space having either a metric or a symplectic structure; either way the product of any two such elements is an ordinary complex number. This product does not satisfy the associative law.

Before we begin with the presentation of this approach, a brief excursion into the structure of the super-symmetry group is in order. This group consists of Lorentz rotations and translations of space–time and of the super-gauge transformations. This group has a homomorphism with the Lorentz rotations alone, the normal subgroup including both translations and super-gauge transformations. This normal subgroup again has a normal subgroup of its own, the translations of space–time. The corresponding factor group can be conveniently indexed in terms of the same Majorana spinor θ that is used to describe the super-gauge transformations. This factor group is, however, Abelian (because the translations have been factored out) and in fact isomorphic with addition in the linear vector space of the Majorana spinors, $\alpha\theta + \beta\theta'$, where α, β are complex numbers.

2. FERMION NUMBERS AND SUPER-GAUGE TRANSFORMATIONS

Majorana spinors are four-component spinors subject to algebraic conditions that reduce them essentially to two-component (Van der Waerden) spinors. In order to avoid the need to keep track of the algebraic conditions a two-component notation will be used in all that follows. Any of the equations and results can, of course, also be formulated in four-component spinor notation, as long as it is understood that all spinors occurring are Majorana spinors. In the two-component notation the alternator ($= i\sigma_2$) will be denoted by the symbol ϵ, whose two indices may be both subscripts or superscripts, dotted or undotted, as the occasion may demand.

The components of all spinors will be taken from a linear N-dimensional vector space over the field of complex numbers, whose elements will be identified with those of its dual space through a reversible linear mapping, represented by a real matrix m. Only two classes of mappings will be considered, those in which m is symmetric and those in which m is antisymmetric. The first case includes the possibility that $N = 1$; this is the case in which the spinor components are ordinary complex numbers. In the second case N must be even, and we have a symplectic structure. In the first case m can be diagonalized by an appropriate choice of the base. If m is antisymmetric, it can be given the canonical symplectic form. Whether m is symmetric or antisymmetric, its form will not be changed by real orthogonal or real symplectic substitutions, respectively. Infinitesimal substitutions b of either kind are characterized by the condition

$$mb + b^T m = 0 \tag{2.1}$$

In all that follows, the bilinear form $q^T mp$ will be written more briefly as qp. The two cases indicated above imply either that $qp = pq$ or that $qp = -pq$.

Consider a space coordinatized by four real coordinates x^μ and a spinor θ, with components defined on a given linear vector space over the field of complex numbers. We shall characterize an infinitesimal super-gauge transformation with descriptor η by the mappings that it causes among the left cosets with respect to Lorentz rotations, following the procedure of Salam and Strathdee (1974). If m is symmetric, we postulate

$$\delta\theta = \eta \qquad \delta x^\mu = (i/2)(\eta^\dagger \sigma^\mu \theta - \eta^T \overline{\sigma^\mu \bar{\theta}}) \tag{2.2a}$$

It follows that the commutator of two such transformations, with descriptors η_1 and η_2, is

$$\delta_c\theta = 0 \qquad \delta_c x^\mu = i(\eta_1^\dagger \sigma^\mu \eta_2 - \eta_1{}^T \overline{\sigma^\mu} \bar{\eta}_2) \tag{2.2b}$$

In the antisymmetric case the corresponding equations are

$$\delta\theta = \eta \qquad \delta x^\mu = \tfrac{1}{2}(\eta^\dagger \sigma^\mu \theta + \eta^T \overline{\sigma^\mu \bar{\theta}}) \tag{2.3a}$$

$$\delta_c\theta = 0 \qquad \delta_c x^\mu = \eta_1^\dagger \sigma^\mu \eta_2 + \eta_1{}^T \overline{\sigma^\mu} \bar{\eta}_2 \tag{2.3b}$$

In either case the commutators change signs if the subscripts 1 and 2 are interchanged, as required. We emphasize that all the expressions for δx^μ, $\delta_c x^\mu$ are ordinary numbers and real.[2]

Exponentiation of the infinitesimal mappings (2.2) and (2.3), respectively, is trivial, as the second-order terms in the exponential expansion are already zero. Accordingly, the construction of products of finite transformations and of commutators is straightforward. The details are not very interesting. It might be remarked, parenthetically, that the substitutions in the number space, (2.1), commute both with the super-gauge transformations and with the Poincaré transformations of space–time. The total symmetry group in either case (2.2) or (2.3) is thus fairly involved, but is in any case a finite-dimensional Lie group.

3. SYMMETRY TRANSFORMATIONS OF THE FIRST KIND

Assuming a flat Minkowski space–time, the group of symmetry transformations consists of Poincaré transformations [with associated transforma-

[2] The dagger indicates Hermitian conjugation, T transposition, and the bar conjugate complex.

tions of the group $SL(2, C)$ chosen so as to leave the spin matrices σ^μ form invariant], of super-gauge transformations of type (2.2) or (2.3), respectively, and of substitutions in the number space of type (2.1). The spinor η, which is used to index super-gauge transformations, is constant. The product of "fermion numbers" is not associative, as the product does not lie in the vector space, but in the field of ordinary complex numbers. The only exception is the one-dimensional symmetric case; with $N = 1$ all fermion numbers are ordinary complex numbers.

For the construction of super fields the algebraic properties of the fermion numbers are of great importance. Some of the rules of procedure in the formalism investigated here resemble those applying to the elements of a Grassmann algebra. Depending on whether m is assumed symmetric or antisymmetric, we have, for instance,

$$\eta_1{}^T\epsilon\eta_2 = \pm\eta_2{}^T\epsilon\eta_1 \qquad \eta_1^\dagger\sigma^\mu\eta_2 = \pm\overline{\eta_2^\dagger\sigma^\mu\eta_1} \tag{3.1}$$

the plus corresponding to symmetric, the minus sign to antisymmetric m. One rule, which was of great value in the case of Grassmann elements, as well as with ordinary numbers, is the quartic "rearrangement" formula,

$$(\alpha^\dagger\epsilon\bar{\beta})(\gamma^T\epsilon\delta) = \pm\eta_{\mu\nu}(\alpha^\dagger\sigma^\mu\gamma)(\beta^\dagger\sigma^\nu\delta) \tag{3.2}$$

This formula does not hold in our present situation, as the components of the four spinors involved are not elements of an associative algebra.

Because of the unavailability of equation (3.2) all attempts to construct super fields (i.e., fields over the flat space–time manifold with arbitrary x dependence that would transform together and whose transformation law would be a representation of the complete symmetry group) have been unsuccessful. Such fields will be constructed over curved space–time manifolds in Section 4, by decoupling the super symmetry from the space–time diffeomorphisms. There exist, however, representations with constant components. One representation, which is not faithful, consists of a constant vector u^μ and a constant spinor λ, obeying the following transformation law under super-gauge transformations:

$$\delta\lambda = 0 \qquad \delta u^\mu = i(\eta^\dagger\sigma^\mu\lambda - \eta^T\overline{\sigma^\mu}\bar{\lambda}) \tag{3.3}$$

and

$$\delta\lambda = 0 \qquad \delta u^\mu = \eta^\dagger\sigma^\mu\lambda + \eta^T\overline{\sigma^\mu}\bar{\lambda} \tag{3.4}$$

respectively, depending on whether m is chosen symmetric or antisymmetric. Equations (3.3) and (3.4) can be obtained from the notion of a "super space" as used by Salam and Strathdee (1974). The equations (2.2) and (2.3), respectively, define an infinitesimal mapping of super space on itself. The commutator of two such infinitesimal mappings may be interpreted as the Lie derivative of one "super-vector" field with respect to the other, or as the infinitesimal transformation law of one of these fields under the infinitesimal mapping described by the other. This latter interpretation results in equations (3.3) and (3.4), respectively.

Other representations, also with constant components, can be obtained by the following maneuver. Consider a scalar or tensor field over super space whose functional dependence on both the space–time and the spinor coordinates is a polynomial of degree P. The coefficients of the various products of the coordinates will be constants; under Poincaré transformations these components will transform as components of scalars, tensors, spinors, and mixed quantities. As an example, consider a scalar U whose functional dependence on all coordinates is no higher than the first power. It may be written in the form

$$U = a + b_\mu x^\mu + \lambda^T \epsilon \theta + \lambda^\dagger \epsilon \bar{\theta} \tag{3.5}$$

Under an infinitesimal transformation (2.2) or (2.3) this scalar will undergo a change in its formal dependence on the coordinates x^μ and θ, so that

$$\begin{aligned} \delta a &= -(\lambda^T \epsilon \eta + \lambda^\dagger \epsilon \bar{\eta}) \qquad \delta b_\mu = 0 \\ \delta \lambda &= -(i/2) \epsilon \overline{\sigma^\mu} \bar{\eta} b_\mu \qquad \delta_c a = i b_\mu (\eta_2^\dagger \sigma^\mu \eta_1 - \eta_1^\dagger \sigma^\mu \eta_2) \end{aligned} \tag{3.6}$$

for symmetric m, whereas for antisymmetric m one has

$$\begin{aligned} \delta a &= -(\lambda^T \epsilon \eta + \lambda^\dagger \epsilon \bar{\eta}) \qquad \delta b_\mu = 0 \\ \delta \lambda &= \tfrac{1}{2} \epsilon \overline{\sigma^\mu} \bar{\eta} b_\mu \qquad \delta_c a = b_\mu (\eta_2^\dagger \sigma^\mu \eta_1 - \eta_1^\dagger \sigma^\mu \eta_2) \end{aligned} \tag{3.7}$$

Equations (3.6) and (3.7) are faithful representations of their respective symmetry groups.

As a third example of a representation by constant components take a covariant vector in super space, v. Its components may be designated by the symbols v_μ, $\nu^T \epsilon$, with ν a spinor. The transformation law will be

$$\delta v_\mu = 0 \qquad \delta \nu = -i \epsilon \overline{\sigma^\mu} \bar{\eta} v_\mu \tag{3.8}$$

or

$$\delta \nu = -\epsilon \overline{\sigma^\mu} \bar{\eta} v_\mu$$

the first expression again corresponding to symmetric m, the second to antisymmetric m. The scalar product of the super multiplet (3.3), (3.4), and the expressions (3.8) is

$$v \cdot u = v_\mu u^\mu + \nu^T \epsilon \lambda + \nu^\dagger \epsilon \bar{\lambda} \tag{3.9}$$

and is super-gauge invariant.

4. SYMMETRY TRANSFORMATIONS OF THE SECOND KIND

In the previous section we had considered super-gauge transformations grafted on the Poincaré group. For the commutator of two such transformations to lie in the Poincaré group the super-gauge transformations themselves had to be characterized by a finite number of (constant) parameters, i.e., they needed to be transformations of the first kind. That any such transformation group can be generalized into a transformation of the second kind has been demonstrated, e.g., by Yang and Mills (1954). Because of the intimate connection between the super-gauge transformations and the Poincaré group (or some other Lie group such as the conformal group), the transition to super-gauge transformations of the second kind is suggested in particular if the super-gauge transformations are to be tied to the group of diffeomorphic mappings of space–time onto itself, in other words, if we are concerned with "super gravity."

Such a connection can be made if the representations described in Section 3 are tied to individual world points and if the Poincaré group is replaced by tetrad transformations. In the case of the representations (3.3), (3.4), which do not involve a faithful representation of the full symmetry group, one need not consider Poincaré transformations of the tetrad but can confine oneself to tetrad rotations (i.e., Lorentz transformations). In the following, this route will be followed, without foreclosing the consideration of a more elaborate scheme in the future.

In setting up a formalism that will be manifestly covariant one needs to erect at each point of space–time a fiber, which will incorporate all the fields required. These will be the fields λ and u^μ of equations (3.3), (3.4), and in addition some reference to the local tetrads. There are two entirely equivalent formulations; either one can retain the Pauli spin matrices σ^k as fixed arrays of pure numbers and work with the 16 components of four orthonormal world vectors at each point $e_k{}^\mu$, or one can work with the matrix-valued vector field σ^μ, which will be defined as a set of four linearly independent Hermitian 2×2 matrices. In the latter formalism Lorentz rotations are represented by unimodular transformations in the complex two-dimensional spin space attached to each point. Finally, each fiber must permit substitutions in the space of fermion numbers.

In order to have any meaningful analysis, one needs to have structures that define "horizontality," that is to say, those mappings of neighboring fibers on each other that will be considered identity mappings. The notion of horizontality is the straightforward generalization of parallel transfer of a vector from one world point to a neighboring world point. Horizontal displacement in a fiber bundle, just like parallel transfer, need not be integrable: If fibers are mapped horizontally on neighboring fibers all along a closed path in space–time, the mapping of the initial fiber on itself on return to the point of departure is in general not an identity mapping. Any such nonidentical mapping can be represented for the infinitesimal closed path by a two-form, which for obvious reasons is usually referred to as the curvature.

For the fixation of horizontal displacement, separate one-forms are needed to correspond to each kind of symmetry. For substitutions in the fermion number space there will be a real field β_ι,

$$m\beta_\iota + \beta_\iota^T m = 0 \tag{4.1}$$

in analogy to equation (2.1). For defining the horizontality of transport with respect to the complex two-dimensional spin space there will be a connection α_ι,

$$\operatorname{tr} \alpha_\iota = 0 \tag{4.2}$$

There must be a connection γ_ι to compensate for super-gauge transformations. The ordinary affine connection, the Christoffel symbols, is connected with the α_ι so as to render the generalized Pauli matrices σ^μ covariantly constant. All told there are the following rules for horizontal displacement:

$$\begin{aligned} d\sigma^\mu &= \left[-\begin{Bmatrix}\mu\\ \nu\iota\end{Bmatrix}\sigma^\nu + \alpha_\iota^\dagger\sigma^\mu + \sigma^\mu\alpha_\iota\right] dx^\iota = \sigma^\mu_{\ ,\iota}\, dx^\iota \\ d\lambda &= -(\alpha_\iota + \beta_\iota)\lambda\, dx^\iota \\ du^\mu &= \left[-\begin{Bmatrix}\mu\\ \nu\iota\end{Bmatrix}u^\nu - i(\gamma_\iota^\dagger\sigma^\mu\lambda - \gamma_\iota{}^T\overline{\sigma^\mu}\bar{\lambda})\right] dx^\iota \end{aligned} \tag{4.3}$$

or

$$du^\mu = \left[-\begin{Bmatrix}\mu\\ \nu\iota\end{Bmatrix}u^\nu - (\gamma_\iota^\dagger\sigma^\mu\lambda + \gamma_\iota{}^T\overline{\sigma^\mu}\bar{\lambda})\right] dx_\iota$$

depending on whether m is symmetric or antisymmetric. The corresponding commutators will map each fiber on itself:

$$\begin{aligned} d\sigma^\mu &= (P_{\iota\kappa}^\dagger\sigma^\mu + \sigma^\mu P_{\iota\kappa} - R^\mu_{\iota\kappa\lambda}\sigma^\lambda)\, dx^\iota \wedge dx^\kappa = 0 \\ d\lambda &= -(B_{\iota\kappa} + P_{\iota\kappa})\lambda\, dx^\iota \wedge dx^\kappa \qquad mB_{\iota\kappa} + B_{\iota\kappa}^T m = 0 \\ du^\mu &= -(R^\mu_{\iota\kappa\lambda}u^\lambda + \Gamma_{\iota\kappa}^\dagger\sigma^\mu\lambda + \Gamma_{\iota\kappa}^T\overline{\sigma^\mu}\bar{\lambda})\, dx^\iota \wedge dx^\kappa \end{aligned} \tag{4.4}$$

$R^\mu_{\iota\kappa\lambda}$ is the customary Riemann–Christoffel tensor that characterizes the metric curvature of space–time. $P_{\iota\kappa}$ is the spin curvature tensor, algebraically related to the Riemann–Christoffel tensor. The real field $B_{\iota\kappa}$ represents the nonintegrability of the structure m, which relates the elements of the fermion number space to those of the dual space. Finally, $\Gamma_{\iota\kappa}$ relates to the super-gauge transformations. The expression given above assumes that m is antisymmetric. With symmetric m, the two terms involving $\Gamma_{\iota\kappa}$, in the last line, would have to have opposite signs and be multiplied by i.

$\Gamma_{\iota\kappa}$ is a spinor-valued two-form. It obeys the (infinitesimal) transformation law

$$\delta\Gamma_{\iota\kappa} = -(P_{\iota\kappa} + B_{\iota\kappa})\eta \tag{4.5}$$

which does not involve derivatives of η; the transformation law of γ_ι does. One cannot construct super-gauge-invariant expressions from $\Gamma_{\iota\kappa}$ alone. This is possible by introducing the super field (3.8). The product of the components v_μ, $\nu^T\epsilon$ by the right-hand sides of equations (4.4) (second line) will be super-gauge invariant.

There are other methods that also lead to super-gauge-invariant vectors and tensors. With the help of the expressions (4.3) one can form the analog of covariant derivatives of super fields, e.g., $u_{;\iota}$, which under super-gauge transformations obey a transformation law exactly analogous to that of the super field u itself. Its product by v will again be super-gauge invariant. From all of these expressions one can construct Lagrangians, for the formalism contemplated here involves a pseudo-Riemannian metric in space–time, determined algebraically and uniquely by the field σ^μ. This metric tensor is given by the expression

$$g^{\mu\nu} = -\tfrac{1}{2}(\sigma^\mu\epsilon\bar{\sigma}^\nu + \sigma^\nu\epsilon\bar{\sigma}^\mu)\epsilon \tag{4.6}$$

A possible term in a Lagrangian density is, for instance,

$$L = (|g|)^{1/2}g^{\mu\nu}u_{;\mu}\cdot v_{;\nu} \tag{4.7}$$

where u, v and the connecting dot have the significance of equation (3.9). A differently structured term in a Lagrangian density may be obtained by multiplying the super-curvature tensor (4.4) by u and by v, and by squaring the resulting super-scalar-valued two-form. Thus the opportunities for constructing super-gauge-invariant scalar densities are almost unlimited, and one will have to rely on physical motivations to make a selection.

5. CONCLUDING REMARKS

The purpose of this paper has been to demonstrate that there are mathematically well-defined quantities that permit the realization of gauge groups by transformations that mix quantities having half-odd spin with those having integral spin. There are mappings of super space (the topological product of space–time by a linear vector space) on itself which have the Poincaré group or the Lorentz group as a subgroup and which resemble closely the groups investigated by Salam and Strathdee (1974). Indeed our structure groups need not differ from those found in the literature. The thrust of our paper is to call attention to group realizations more general than those in the literature.

Any of these constructions based on a flat space–time can be utilized to form a fiber bundle that admits general diffeomorphisms of its base manifold, space–time, and a super-symmetry group of its fibers. It is probably possible to generate structures with a super-gauge-sensitive metric field as well, but this possibility has not yet been investigated. It appears certain, though, that there is a wealth of formal possibilities that resemble theories involving Grassmann elements but that avoid some of the latter's conceptual difficulties.

REFERENCES

Corwin, L., Ne′eman, Y., and Sternberg, S. (1975). *Reviews of Modern Physics*, **47**, 573. This review paper contains numerous references to the literature prior to 1975.

Salam, A., and Strathdee, J. (1974). *Nuclear Physics*, **B76**, 477.

Yang, C. N., and Mills, R. L. (1954). *Physical Review*, **96**, 191.